Solution of System of Linear Equations

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Overview

A linear equation system is a set of linear equations to be solved simultanously.

A linear equation takes the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the $n+1$ coefficients $a_0, a_1, \ldots, a_n, b$ are constants and $x_1, \ldots, x_n$ are the n unknowns.

Following the notation above, a system of linear equations is denoted as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Overview

This system consists of m linear equations, each with $n+1$ coefficients, and has n unknowns which have to fulfill the set of equations simultanously. To simplify notation, it is possible to rewrite the above equations in matrix notation:

$$Ax = b.$$

The elements of the $m \times n$ matrix A are the coefficients of the equations, a_{ij} and the vectors x and b have the elements x_i and b_i respectively. In this notation each line forms a linear equation.

We discuss some direct and iterative methods in three lectures for finding the solution of system of linear systems.

An Example

A shopkeeper offers two standard packets because he is convinced that north indians each more wheat than rice and south indians each more rice than wheat. Packet one P_1 : 5kg wheat and 2kg rice ; Packet two P_2 : 2kg wheat and 5kg rice. Notation. (m, n) : m kg wheat and n kg rice.

Suppose I need 19kg of wheat and 16kg of rice. Then I need to buy x packets of P_1 and y packets of P_2 so that $x(5, 2) + y(2, 5) = (10, 16)$.

Hence I have to solve the following system of linear equations

$$\begin{aligned} 5x + 2y &= 10 \\ 2x + 5y &= 16. \end{aligned}$$

Suppose I need 34 kg of wheat and 1 kg of rice. Then I must buy 8 packets of P_1 and -3 packets of P_2. What does this mean? I buy 8 packets of P_1 and from these I make three packets of P_2 and give them back to the shopkeeper.

Simple example

The solution of the equation

$$ax + by = c$$

is the set of all points satisfy the equation forms a straight line in the plane through the point $(c/b, 0)$ and with slope $-a/b$.

Two lines (parallel - no solution); (intersect - unique solution); (same - infinitely many solutions).

The $m \times n$ matrix A is called the **coefficient matrix**. The $m \times (n+1)$ matrix given by

$$(A, B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

is called the **augmented matrix** of the system. A solution is a set of numbers $x_1, x_2, \ldots, x_n$ which satisfy all the m equations.

A given system of linear equations can be solved by Cramer's rule or by matrix methods. But this method becomes tedious for large systems. Hence we develop several numerical methods such as **Gauss elimination method, Gauss Jordan method, Crout's method, Jacobi method, Seidel method and Relaxation method** for finding the solutions which are well suited for implementation in computers.

Some methods yield the required result after some computations which can be specified in advance. Such methods are called **direct methods**.

An **iterative method** is one in which we start from an approximation to the actual solution and obtain better and better approximation from a computational cycle which is repeated sufficiently many number of times to get the solution to the desired accuracy.

Simultaneous Equations : Back Substitution

Consider a system of simultaneous linear equations given by $AX = B$ where A is an $n \times n$ coefficient matrix. Suppose the matrix A is **upper triangular**.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{mn} \end{pmatrix}.$$

Then the given system takes the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

$$\text{(i.e.)} \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \qquad \cdots \qquad \cdots \qquad & \cdots \\ a_{n-1n-1}x_{n-1} + a_{n-1}x_n &= b_{n-1} \\ a_{nn}x_n &= b_n. \end{aligned}$$

From the last equation we get $x_n = \frac{b_n}{a_{nn}}$.

Substituting the value of x_n in the previous equation we get

$$x_{n-1} = \frac{1}{a_{n-1n-1}}\left[b_{n-1} - a_{n-1}\left(\frac{b_n}{a_{nn}}\right)\right].$$

Proceeding like this we can find all x_i's. This procedure is known as **back substitution**. Similarity considering lower triangular matrix

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

the given system takes the form

$$\begin{aligned} a_{11}x_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \\ \cdots \quad \cdots \quad & \cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

Simultaneous Equations

From the first equation we get $x_1 = \frac{b_1}{a_{11}}$. Substituting the value x_1 in the next equation we can get

$$x_2 = \frac{1}{a_{22}} \left[b_2 - a_{21} \left(\frac{b_1}{a_{11}} \right) \right].$$

Proceeding like this we can find all x_i's. This procedure is known as **forward substitution**.

Gauss Elimination Method

Gauss elimination method is a direct method which consists of transforming the given system of simultaneous equations to an equivalent **upper triangular system**.

From this transformed system the required solution can be obtained by the method of back substitution. Consider the system of n equations in n unknowns given by $AX = B$ where A is the coefficient matrix.
The augmented matrix is

$$(A, B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array} \right).$$

To transform the system to an equivalent upper triangular system, we use the following row operations.

1. The row operation $R_i \to R_i - \frac{a_{i1}}{a_{11}} R_1$; $i = 1, 2, 3, \ldots, n$ makes all the entries $a_{21}, a_{31}, \ldots, a_{n1}$ in the first column zero. Here the first equation is the **pivotal equation**. $a_{11} \neq 0$ is called **pivot** and $\frac{-a_{i1}}{a_{11}}$ for $i = 2, 3, \ldots$ are called **multipliers** for first elimination. If $a_{11} = 0$, we interchange the first row with another suitable row so as to have $a_{11} \neq 0$.
2. Next we do the row operation $R_i \to R_i - \frac{a_{2i}}{a_{22}} R_2$; $i = 3, 4, \ldots, n$. This makes all entries $a_{32}, a_{42}, \ldots, a_{n2}$ in the second column zero.
3. In general the row operation

$$R_i \to R_{i-1} - \frac{a_{ik}}{a_{kk}} R_k; i = k+1, k+2, \ldots, n$$

will make all the entries $a_{k+1,k}, a_{k+2,k}, \ldots, a_{nk}$ in the k^{th} column zero.

Gauss-Jordan elimination method

Hence the given system of equations is reduced to the form $UX = D$ where U is an upper triangular matrix. The required solution can be obtained by the method of **back substitution**.

Consider the system of equations $AX = B$. If A is a **diagonal matrix** the given system reduces to

$$\begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & a_{22} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

This system reduces to the following n equations $a_{11}x_1 = b_1$; $a_{22}x_2 = b_2$; $a_{nn}x_n = b_n$. Hence we get the solution directly as $x_1 = \frac{b_1}{a_{11}}$; $x_2 = \frac{b_2}{b_{22}} \cdots x_n = \frac{b_n}{a_{nn}}$. The method of obtaining the solution of the system of equations by reducing the matrix A to a diagonal matrix is known as **Gauss-Jordan elimination** method.

Calculation of inverse of a matrix

Let A be an $n \times n$ non-singular matrix. Let

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

be the inverse of A.

$\therefore AX = I$ where I is the unit matrix of order n.

$\therefore AX = I$ gives

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Calculation of inverse of a matrix

This equation is equivalent to the following n system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots$$

$$\cdots\cdots \qquad \cdots\cdots \qquad \cdots\cdots$$

Calculation of inverse of a matrix

And

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Each of the system of the above n systems of equations can be solved by Gauss elimination method or Gauss Jordan method.

Exercises

1. Solve the following system of equations using Gaussian elimination method.

$$
\begin{aligned}
x + y + z &= 9 \\
2x - 3y + 4z &= 13 \\
3x + 4y + 5z &= 40.
\end{aligned}
$$

2. Solve the following equations by Gauss Jordan method.

$$
\begin{aligned}
x + y &= 2 \\
2x + 3y &= 5.
\end{aligned}
$$

Exercises

1. Find the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$ using Gaussian method.
2. Solve the following system of equations by Gauss Jordan method

$$\begin{aligned} 5x - 2y + 3z &= 18 \\ x + 7y - 3z &= -22 \\ 2x - y + 6z &= 22. \end{aligned}$$

Consider the system of n equations given by

$$AX = B \tag{1}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n. \end{pmatrix}$$

Crout's method of solving (1) is a direct method which involves the following steps.

Step 1. The matrix A is expressed in the form

$$A = LU \tag{2}$$

where L is a lower triangular matrix given by

$$L = \begin{pmatrix} \ell_{11} & 0 & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \dots & \dots \\ \ell_{n1} & \ell_{n2} & \cdots & \cdots & \ell_{nn}. \end{pmatrix}$$

and U is an upper triangular matrix given by

$$U = \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Step 2. From $A = LU$, L and U can be obtained by equating the corresponding elements of matrices on both sides.
Step 3. Now using (2) in (1) we get

$$LUX = B \tag{3}$$

Let $UX = B'$ where $B' = \begin{pmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{pmatrix}$

Then (3) becomes $LB' = B$.
Step 4. From $LB' = B$ we can obtain B' by **forward substitution**.
Step 5. From $UX = B'$, X can be found by **backward substitution**.

Crout's method

Crout's method can be used to find the inverse of a non-singular matrix.

Since $A = LU$, we get $A^{-1} = U^{-1}L^{-1}$.

Note that inverse of an upper (lower) triangular matrix is an upper (lower) triangular matrix.

Exercises

1. Solve the following equations by Crout's method

$$x + y = 2$$
$$2x + 3y = 5$$

2. Solve the following equations by Crout's method.
   $x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$.
3. Solve the following system of equations by Crout's method
   $2x - 6y + 8z = 24$; $5x + 4y - 3z = 2$; $3x + y + 2z = 16$.
4. Find the inverse of the matrix by Crout's method

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}.$$

Consistency of System of Linear Equations

Consider the system of m linear equations in n unkowns

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots \quad + \quad \vdots \quad + \cdots + \quad \vdots \quad & = & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m. \end{array}$$

Its matrix form is

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

That is, $Ax = b$, where A is called the **coefficient matrix**.

To determine whether these equations are consistent or not, we find the ranks of the **coefficient matrix** A and the **augmented matrix**

$$K = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix} = [A \quad b].$$

We denote the **rank** of A by $r(A)$.

1. $r(A) \neq r(K)$, then the linear system $Ax = b$ is **inconsistent** and has **no solution**.
2. $r(A) = r(K) = n$, then the linear system $Ax = b$ is **consistent** and has a **unique solution**.
3. $r(A) = r(K) < n$, then the linear system $Ax = b$ is **consistent** and has an **infinite number of solutions**.

Solution of Linear Simultaneous Equations

Simultaneous linear equations occur quite often in engineering and science. The analysis of electronic circuits consisting of invariant elements, analysis of a network under sinusoidal steady-state conditions, determination of the output of a chemical plant, finding the cost of chemical reactions are some of the problems which depend on the solution of simultaneous linear algebraic equations, the solution of such equations can be obtained by **direct** or **iterative methods**. Some direct methods are as follows:

1. Method of determinants – Cramer's rule
2. Matrix inversion method
3. Gauss elimination method
4. Gauss-Jordan method
5. Factorization (triangulization) method.

Iterative Methods

Direct methods yield the solution after a certain amount of computation.

On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy.

Thus in an iterative method, the amount of computation depends on the degree of accuracy required.

For **large systems**, iterative methods may be faster than the direct methods. Even the round-off errors in iterative methods are smaller. In fact, iteration is a self correcting process and any error made at any stage of computation gets automatically corrected in the subsequent steps.

Iterative Methods

We now discuss some iterative methods. However iterative methods can be applied only when each equation of the system has one coefficient which is sufficiently large and the large coefficient must be attached to a different unknown in the system. We state the above condition in a more precise form in the following definition.

Definition

An $n \times n$ matrix A is said to be **diagonally dominant** *if the absolute value of each leading diagonal element is greater than or equal to the sum of the absolute values of the remaining elements in that row.*

For example $A = \begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$ is a diagonally dominant matrix and $B = \begin{pmatrix} 2 & 3 & -1 \\ 5 & 8 & -4 \\ 1 & 1 & 1 \end{pmatrix}$ is not a diagonally domiant matrix.

In the system of simultaneous linear equations in n unknowns $AX = B$ if A is diagonally dominant, then the system is said to be a **diagonal system.**

Thus the system equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3z + b_3y + c_3z &= d_3 \end{aligned}$$

is a diagonal system if

$$\begin{aligned} |a_1| &\geq |b_1| + |c_1| \\ |b_2| &\geq |a_2| + |c_2| \\ \text{and } |c_3| &\geq |a_3| + |b_3|. \end{aligned}$$

Exercises

Exercise

1. *Check whether the system of equations*

$$\begin{aligned} x + 6y - 2z &= 5 \\ 4x + y + z &= 6 \\ -3z + y + 7z &= 5 \end{aligned}$$

   is a diagonal system. If it is not, make it a diagonal system.

2. *Is the system of equations diagonally dominant? If it is not, make it diagonally dominant.*
   $3x + 9y - 2z = 10; \quad 4x + 2y + 13z = 19; \quad 4z - 2y + z = 3.$

Gauss-Jacobi Iteration Method

The process of iteration in solving $AX = B$ will converge quickly if the coefficient matrix A is diagonally dominant.

If the coefficient matrix is not diagonally dominant, we must rearrange the equations in such a way that the resulting coefficient matrix becomes dominant, if possible, before we apply the iteration method.

We now discuss Gauss-Jacobi iteration method (or, simply Jacobi's iterative method).

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ x_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ \cdots \qquad \cdots \qquad \cdots \qquad &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= c_n. \end{aligned}$$

Gauss-Jacobi Iteration Method

We assume that the coefficient matrix of this system is diagonally dominant. The above equations can be rewritten as

$$x_1 = \frac{1}{a_{11}}(c_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \qquad -(1)$$

$$x_2 = \frac{1}{a_{22}}(c_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \qquad -(2)$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$x_n = \frac{1}{a_{nn}}(c_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}\, x_{n-1}. \qquad -(n)$$

We start with the initial values for the variables $x_1, x_2, x_3, \dots, x_n$ to be $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots, x_n^{(0)}$.

Using these values in $(1), (2), \cdots, (n)$ respectively we get $x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(0)}$.

Putting $x_1 = x_1^{(1)}, x_2 = x_2^{(1)}, \ldots, x_n = x_n^{(1)}$ in $(1), (2), \ldots, (n)$ respectively we get the next approximations $x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}$.

In general if the values of $x_1, x_2, \ldots, x_n$ in the r^{th} iteration are $x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)}$. then

$$x_1^{(r+1)} = \frac{1}{a_{11}}\left[c_1 - a_{12}x_1^{(r)} - a_{13}x_2^{(r)} - \cdots - a_{1n}x_n^{(r)}\right]$$
$$x_2^{(r+1)} = \frac{1}{a_{22}}\left[c_2 - a_{21}x_1^{(r)} - a_{23}x_2^{(r)} - \cdots - a_{1n}x_n^{(r)}\right]$$
$$x_n^{(r+1)} = \frac{1}{a_{nn}}\left[c_n - a_{n1}x_1^{(r)} - a_{n2}x_2^{(r)} - \cdots - a_{n,n-1}x_n^{(r)}\right].$$

In solving a specific problem in the absence of any specific initial values for the variables we usually take the initial values of the variables to be zero.

Gauss-Jacobi Iteration Method

Consider the system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3. \end{aligned}$$

If a_1, b_2, c_3 are large as compared to other coefficients (if not, rearrange the given linear equations), solve for x, y, z respectively.

Then the system can be written as

$$\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y). \end{aligned} \tag{4}$$

Let us start with the initital approximations x_0, y_0, z_0 for the values of x, y, z repectively. In the absence of any better estimates for x_0, y_0, z_0, these may each be taken as zero.

Substituting these on the right sides of (4), we gt the first approximations.

This process is repeated till the difference between two consecutive approximations is negligible.

Exercises

1. Solve, by Jacobi's iterative method, the equations

$$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25. \end{aligned}$$

2. Solve, by Jacobi's iterative method correct to 2 decimal places, the equations

$$\begin{aligned} 10x + y - z &= 11.19 \\ x + 10y + z &= 28.08 \\ -x + y + 10z &= 35.61. \end{aligned}$$

3. Solve the equations, by Gauss-Jacobi iterative method

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9. \end{aligned}$$

4. Solve the following equations using Jacobi's iteration method.
   $3x + 4y + 15z = 54.8; \quad x + 12y + 3z = 39.66; \quad 10x + y - 2z = 7.74.$

Gauss-Seidel iteration method is a refinement of Gauss-Jacobi method. As in the Jacobi iteration method let

$$x_1 = \frac{1}{a_{11}}(c_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \qquad -(1)$$

$$x_2 = \frac{1}{a_{22}}(c_2 - a_{12}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \qquad -(2)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$x_n = \frac{1}{a_{nn}}(c_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}\,x_{n-1}). \qquad -(n)$$

We start with the initial values $x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)}$ and we get from (1)

$$x_1^{(1)} = \tfrac{1}{a_{11}}\left[c_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \cdots - a_{1n}x_n^{(0)}\right].$$

Gauss-Seidel Iteration Method

In the second equation we use $x_1^{(1)}$ for x_1 and $x_3^{(0)}$ for x_3 etc. and $x_n^{(0)}$ for x_n. (In the Jacobi Method we used $x_1^{(0)}$ for x_1). Thus we get

$$x_2^{(1)} = \tfrac{1}{a_{22}}\big[c_2 - a_{11}x_1^{(1)} - a_{13}x_3^{(0)} - \cdots - a_{2n}x_n^{(0)}\big].$$

Proceeding like this we find the first iteration values as $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$. In general if the values of the variables in the r^{th} iteration are $x_1^{(r)}, x_2^{(r)}, \ldots, x_n^{(r)}$ then the values in the $(r+1)^{th}$ iteration are given by

$$\begin{aligned}
x_1^{(r+1)} &= \frac{1}{a_{11}}\big[c_1 - a_{12}x_2^{(r)} - a_{13}x_3^{(r)} - \cdots - a_{1n}x_n^{(r)}\big] \\
x_2^{(r+1)} &= \frac{1}{a_{22}}\big[c_2 - a_{11}x_1^{(r+1)} - a_{13}x_3^{(r)} - \cdots - a_{1n}x_n^{(r)}\big] \\
x_n^{(r+1)} &= \frac{1}{a_{nn}}\big[c_n - a_{n1}x_2^{(r+1)} - a_{n2}x_2^{(r+1)} - \cdots - a_{n,n-1}x_{n-1}^{(r+1)}\big].
\end{aligned}$$

Gauss-Seidel Iteration Method

This is a modification of **Jacobi's Iterative Method**.

Consider the system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3. \end{aligned}$$

If a_1, b_2, c_3 are large as compared to other coefficients (if not, rearrange the given linear equations), solve for x, y, z respectively.

Then the system can be written as

$$\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1y - c_1z) \\ y &= \frac{1}{b_2}(d_2 - a_2x - c_2z) \qquad (5) \\ z &= \frac{1}{c_3}(d_3 - a_3x - b_3y). \end{aligned}$$

Here also we start with the initial approximations x_0, y_0, z_0 for x, y, z respectively, (each may be taken as zero).

Subsituting $y = y_0, z = z_0$ in the first of the equations (5), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0).$$

Then putting $x = x_1, z = z_0$ in the second of the equations (5), we get

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0).$$

Next subsituting $x = x_1, y = y_1$ in the third of the equations (5), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

and so on.

Gauss-Seidel Iteration Method

That is, as soon as a new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is repeatedly till the values of x, y, z are obtained to desired degree of accuracy.

Since the most recent approximations of the unknowns are used which proceeding to the next step, **the convergence in the Gauss-Seidel method is twice as fast as in Gauss-Jacobi method.**

Choice of Initial Approximations for Convergence

Jacobi and Gauss-Seidel methods converge for any choice of the initial approximations if

in each equation of the system, the absolute value of the largest co-efficient is almost equal to the sum of the absolute values of all the remaining coefficients.

(OR)

in at least one equation, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients.

Exercise

1. *Solve* $2x + y = 3 : 2x + 3y = 5$ *by Gauss-Seidel iteration method.*
2. *Solve the following system of equations using Gauss-Seidel iteration method.*
   $6x + 15y + 2z = 72; \quad x + y + 54z = 110; \quad 27z + 6y - z = 85.$
3. *Solve the following system of equations using Gauss Seidel iteration method.*
   $$10x + 2y + z = 9 \quad x + 10y - z = -22 \quad -2x + 3y + 10z = 22.$$
4. *Solve the following system of equations by*
   (a) *Gauss-Seidel method* (b) *Gauss-Jacobi method*

   $28x + 4y - z = 32; \quad x + 3y + 10z = 24; \quad 2x + 17y + 4z = 35.$

Exercises

4. Apply Gauss-Seidel iterative method to solve the equations

$$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25. \end{aligned}$$

5. Solve the equations, by Gauss-Jacobi and Gauss-Seidel methods (and compare the values)

$$\begin{aligned} 27x + 6y - z &= 85 \\ x + y + 54z &= 110 \\ 6x + 15y + 2z &= 72. \end{aligned}$$

6. Apply Gauss-Seidel iterative method to solve the equations

$$\begin{aligned} 10x_1 - 2x_2 - x_3 - x_4 &= 3 \\ -2x_1 + 10x_2 - x_3 - x_4 &= 15 \\ -x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\ -x_1 - x_2 - 2x_3 + 10x_4 &= -9. \end{aligned}$$

Relaxation Method

We describe this method only for a system of three equations in three unknowns given by

$$
\begin{aligned}
a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2 \\
a_3x + b_3y + c_3z &= d_3
\end{aligned}
$$

We define the residuals

$$
\begin{aligned}
R_x &= a_1x + b_1y + c_1z - d_1 \\
R_y &= a_2x + b_2y + c_2z - d_2 \\
R_z &= a_3x + b_3y + c_3z - d3
\end{aligned}
\tag{1}
$$

We observe that for actual solution of the system the residuals become zero.

Relaxation Method

The relaxation method consists of reducing the values of the residuals as close to zero as possible by modifying the values of the variables at each stage. For this purpose we give an operation table from which we can know the changes in R_x, R_y, R_z corresponding to any change in the values of the variables.

The operation table is given by

Δ x	Δ y	Δ z	ΔR_x	ΔR_y	ΔR_z
1	0	0	$-a_1$	$-a_2$	$-a_3$
0	1	0	$-b_1$	$-b_2$	$-b_3$
0	0	1	$-c_1$	$-c_2$	$-c_3$

The table is the negative of transpose of the coefficient matrix.

Relaxation Method

We note from the equations (1) that if x is increased by 1 (keeping y and z constant) R_x, R_y, R_z decrease by a_1, a_2, a_3 respectively. This is shown in the above table along with the effects on the residuals when y and z are given until increments.

At each step the numerically largest residual is reduced to almost zero. To reduce a particular residual the value of the corresponding variable is changed. When all the residuals are reduced to almost zero the increments in x, y, z are added separately to give the desired solution.

$$x = \sum \Delta x; \quad y = \sum \Delta y; \quad z = \sum \Delta z.$$

At each step, the **numerically largest** residual is reduced to almost zero.

How can one reduce a particular residual?

To reduce a particular residual, the value of the corresponding variable is changed.

For example, to reduce R_x by p, x should be increased by p/a_1.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solution.

Verification Process : The residuals **are not all** negligible when the computed values of x, y, z are substituted in (**??**), then there is some mistake and the entire process should be rechecked.

Sufficient Conditions

Relaxation method can be applied successfully only if there is **at least one row** in which diagonal element of the coefficient matrix dominate the other coefficients in the corresponding row.

That is,

$$\begin{aligned} |a_1| &\geq |b_1| + |c_1| \\ |b_2| &\geq |a_2| + |c_2| \\ |c_3| &\geq |a_3| + |b_3| \end{aligned}$$

should be valid for at least one row.

1. Solve by relaxation method, the equations

$$\begin{aligned} 9x - 2y + z &= 50 \\ x + 5y - 3z &= 18 \\ -2x + 2y + 7z &= 19. \end{aligned}$$

2. Solve by relaxation method, the equations

$$\begin{aligned} 10x - 2y - 3z &= 205 \\ -2x + 10y - 2z &= 154 \\ -2x - y + 10z &= 120. \end{aligned}$$

3. Solve the following equations using relaxation method

$$5x - y - z = 3; \quad -x + 10y - 2z = 7; \quad -x - y + 10z = 8.$$

4. Solve the following equations using relaxation method

$$9x - y + 2z = 9; \quad x + 10y - 2z = 15; \quad 2x - 2y - 13z = -17.$$

References

1. **Richard L. Burden and J. Douglas Faires**, **Numerical Analysis - Theory and Applications**, Cengage Learning, Singapore.
2. **Kendall E. Atkinson**, **An Introduction to Numerical Analysis**, Wiley India.
3. **David Kincaid and Ward Cheney**, **Numerical Analysis - Mathematics of Scientific Computing**, American Mathematical Society, Providence, Rhode Island.
4. **S.S. Sastry**, **Introductory Methods of Numerical Analysis**, Fourth Edition, Prentice-Hall, India.